Quantum 2013

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Agustín: Una nueva aprox a los grupos cuánticos.

(Bridgeland Ann. Math 13)

$\mathfrak{g} \longrightarrow U(\mathfrak{g})$ $x \in \mathfrak{g} \to \Delta x = x \otimes 1 + 1 \otimes x$

$\mathfrak{g} \longrightarrow U_t(\mathfrak{g})$, $t \in \mathbb{k}$, $t^n = 1$.
$n \neq 2$

Grupos cuánticos $U_t(\mathfrak{g})$

$A = (a_{ij})_{1 \leq i,j \leq n}$ gen. de Cartan
(sim)

$i \neq j$ $a_{ij} = a_{ji} \in \mathbb{Z}$, $a_{ii} = 2$

$U_t(\mathfrak{g}) = \mathbb{K}\langle E_i, F_i, k_i, k_i^{-1}, i:1 \ldots n \mid$

$k_i k_i^{-1} = 1 = k_i^{-1} k_i \,, \quad [k_i, k_j] = 0$

$k_i E_j = t^{a_{ij}} E_j k_i \,, \quad k_i F_j = t^{-a_{ij}} F_j k_i$

$$[E_i, F_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{t - t^{-1}}$$

$$0 = \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_t E_i^r F_j E_i^{1-a_{ij}-r}$$

(idem con F_i)

$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}$

$\Delta E_i = E_i \otimes 1 + k_i \otimes E_i$

$\Delta F_i = F_i \otimes k_i^{-1} + 1 \otimes F_i$

Subálgebra distinguida ③

$U_t(\mathfrak{n}^+) = \mathbb{K}\langle E_i \mid 1 \leq i \leq n \rangle$

$U_t(\mathfrak{n}^-) = \mathbb{K}\langle F_i \mid i = 1, \dots, n \rangle$

$U_t(\mathfrak{b}^+) = \mathbb{K}\langle E_i, k_j, \mid \dots \rangle$

$U_t(\mathfrak{b}^-) = \mathbb{K}\langle F_i, k_j \mid \dots \rangle$

$U_t(\mathfrak{h}) = \mathbb{K}\langle k_i, k_i^{-1} \mid \dots \rangle$

$\simeq \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm n}]$

Prop (Descomposición triangular)

$\exists$ isom. de ev

$$U_t(\mathfrak{n}^+) \otimes U_t(\mathfrak{h}) \otimes U_t(\mathfrak{n}^-) \xrightarrow{\cong} U_t(\mathfrak{g})$$

$$U_t(\mathfrak{b}^+) \otimes U_t(\mathfrak{n}^-) \longrightarrow U_t(\mathfrak{g})$$

$$U_t(\mathfrak{n}^+) \otimes U_t(\mathfrak{b}^-) \longrightarrow U_t(\mathfrak{g})$$

Más aún están determinados por la mult. en $U_t(\mathfrak{g})$.

$\square$

Ingredientes

1. Álgebras de caminos y sus representaciones
2. Complejos de cadena,

cohomología, extensiones. 5

3. Álgebras de Hall $\mathcal{H}_0(\mathcal{A})$ de una cat. abeliana $\mathcal{A}$

a) $H(\mathrm{Rep}_k Q)$, Q carcaj

$(\simeq U_t(\mathfrak{n}^+))$

b) $H(C_{\mathbb{Z}_2}(\mathrm{Rep}\, kQ))$

$\rightsquigarrow (U_t(\mathfrak{g}))$

Un <u>carcaj</u> es una colección

$(Q_0, Q_1, s, t : Q_1 \longrightarrow Q_1)$

Q_0 es un conjunto finito de "vértices"

Q_1 es un cjto de flechas

$p \in Q_1$, $s(p)$ = comienzo de p

$t(p)$ = final de p

Un camino <u>no</u> trivial es una suc. de flechas $p_1 — p_m$ tales que $s(p_i) = t(p_{i+1})$

Si Q es un carcaj, el álg. de caminos es el e.v.

$k\{x \mid x$ es un camino$\}$ con la mult. dada por (7)

$$x \cdot y = \begin{cases} xy & \text{si } s(x) = t(y) \\ 0 & \text{c.c} \end{cases}$$

<u>Ejemplo</u> $\underset{1}{\circ} \xrightarrow{\sigma} \underset{2}{\circ} \xrightarrow{\tau} \underset{3}{\circ}$

$$kQ = k\{e_1, e_2, e_3, \sigma, \tau, \tau\sigma\}$$

e_i = camino trivial en el vértice i

$\Rightarrow e_i e_j = \delta_{ij} e_i$

$$1 = \sum_{i=1}^{n} e_i$$

$kQ = A = \bigoplus_{i=1}^{n} Ae_i$

e_i idemp. indesc. (primitivos)

$\Rightarrow Ae_i$ mód. proy. indesc.

A de dim finita $\longleftrightarrow$ Q no tiene ciclos orienta-bles

Una representación de Q

es una colección

$((V_i)_{i\in Q_0}, (x_\rho)_{\rho\in Q_1})$ de esp. vect.

y trans. lineales $x_\rho \in \mathrm{Hom}_k(V_{s(\rho)}, V_{t(\rho)})$

Rep $Q \simeq \mathrm{Mod}\, kQ$ 9

$((V_i),(x_\rho)) \rightsquigarrow V = \bigoplus V_i$

$$V_i \xrightarrow{\varepsilon_i} V \xrightarrow{\pi_j} V_j$$

Si x es un camino $x = \rho_1 \cdots \rho_m$

$$x \cdot v = \varepsilon_{t(\rho_1)} x_{\rho_1} \cdots x_{\rho_m} \pi_{s(\rho_m)} v$$

Rep. simples

$i \in Q_0 \longrightarrow \exists\ S(i)$ rep simple

$S(i) = ((V_j), (x_\rho))$

$$V_j = \begin{cases} k & j = i \\ 0 & \text{c.c} \end{cases}$$

$$x_\rho = 0 \quad \forall \rho$$

si Q no tiene ciclos orientados $\Rightarrow$ son todos

Extensiones

si A es una categoría abeliana, $M, N \in \mathrm{Obj}\,A \Rightarrow$

$$\mathrm{Ext}^1_A(M,N) = \left\{ \begin{array}{c} 0 \to N \to E \to M \to 0 \\ \text{EXACTA} \end{array} \right\} \Big/ \sim$$

Dos sus. exactas son equiv. si

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & M & \to & 0 \\ & & \| & \wr\!\!\wr & \downarrow h & \wr\!\!\wr & \| & & \\ 0 & \to & N & \to & E & \to & M & \to & 0 \end{array} \quad (11)$$

$\exists\, h:$

$A = \mathrm{Rep}\, Q$

<u>Ejer</u> $\dim \mathrm{Ext}^1(S_i, S_j) =$

$= \#\{$caminos de i a $j\}$

Álgebras de Hall

Sea A una categoría abel. $\mathbb{F}_q$-lineal, A es esencial //

pequeña, $\dim_{\mathbb{F}_q} \mathrm{Hom}(M,N) < \infty$ 12

$\dim_{\mathbb{F}_q} \mathrm{Ext}^1(M,N) < \infty$

- $\mathcal{A}$ es de dim global finita y tiene sufic. proyectivos.

Sea $\mathrm{Ext}^1_{\mathcal{A}}(M,N)_L \subseteq \mathrm{Ext}^1_{\mathcal{A}}(M,N)$ como el subconjunto

$\{0 \to M \to X \to L \to 0 \mid X \cong L\}$

Sea $\mathrm{Iso}(\mathcal{A}) = \mathrm{obj}\,\mathcal{A}/\sim$

El álgebra de Hall $\mathcal{H}(\mathcal{A})$ es el k-esp. vect de base

$\{[A] \mid [A] \in \mathrm{Iso}\,\mathcal{A}\}$ (13)

$$[A_1] \diamond [A_2] = \sum_{[B] \in \mathrm{Iso}\,\mathcal{A}} \frac{|\mathrm{Ext}^1(A_1, A_2)_B|}{|\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)|} [B]$$

Notar $[0]$ es la unidad

$$\overset{!}{=} \sum_{[B] \in \mathrm{Iso}\,\mathcal{A}} \frac{a_1 a_2 \, g^B_{A_1 A_2}}{b} B$$

$a_i = |\mathrm{Aut}\, A_i|$, $b = |\mathrm{Aut}\, B|$

$$g^B_{A_1, A_2} = |\{C \subseteq B \mid C \cong A_2 \wedge B/C \cong A_1\}|$$

$$A_1 \diamond A_2 \diamond A_3 = \sum g^B_{A_1, A_2, A_3} B$$

$$g^{B}_{A_1,A_2,A_3} = \left\{ X \leq Y \leq B \;\middle|\; \begin{array}{l} X \cong A_3,\ Y/X \cong A_1 \\ B/Y \cong A_2 \end{array} \right\}$$

Si $A_1, A_2 \in \mathcal{A} \rightsquigarrow \langle A_1, A_2 \rangle =$ (14)

$$= \sum (-1)^i \dim \operatorname{Ext}^i(A_1, A_2)$$

$$A_1 * A_2 = t^{\langle A_1, A_2 \rangle} A_1 \diamond A_2$$

<u>Obs</u> si $\mathcal{A}$ es hereditaria

$$\langle A_1, A_2 \rangle = \dim \operatorname{Hom} - \dim \operatorname{Ext}^1$$

$$\mathcal{A},\ U_t(\hat{g}),\ U_t(n^+),\ q \text{ s.t. } t = \sqrt{q}$$

[Ringel, Green, Kreimer] Q carcaj asoc. a Δ

$|Q_0| = n$

$|\{i - j\}| = a_{ij}$

$U_t(n^+)$ es la subálg. de $\mathcal{H}_e(\mathrm{Rep}\,\mathbb{F}_q Q)$ generada por

$[S_i]/_{q-1}$

Más aún, si Q es ADE

$\Rightarrow U_t(n^+) \simeq \mathcal{H}_e(\mathrm{Rep}\,\mathbb{F}_q Q)$

En el caso grad, $\mathcal{A} = C_{\mathbb{Z}/2}(\mathrm{Rep}\,Q)$

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$$\left(M_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} M_0 \,,\; M_i \in \operatorname{Rep} Q \atop d_i d_{i+1} = 0 \right)$$

$$A = \mathbb{C}_{\mathbb{Z}_2}(P)$$

$$D\mathcal{H}(A) = \mathcal{H}(A)\left[[M_\bullet]^{-1} \,/\, H_*(M_\bullet) = 0 \right]$$

$$DH_{red}(A) = DH(A)\left[[M_\bullet]^{-1} \,/\, H_*((M_\bullet)) = 0 \atop M_\bullet \simeq M_\bullet^* \right]$$

Teo (Bridgeland)

$\exists$ un morf iny. de álg.

$$U_t(\mathfrak{g}) \hookrightarrow DH_{red}(A)$$

$\Delta DB \rightarrow$ iso

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(U. Santiago)

Módulos de Sörgel

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$m < \dim_{\mathbb{C}} V = n$

1900 Schur $\quad SL(V) \curvearrowright V^{\otimes m} \curvearrowleft S_m$ 1830

Schur - Weil duality

$\mathrm{End}_{SL(V)}(V^{\otimes m}) \cong \mathbb{C}S_m$

$\mathrm{End}_{\mathbb{C}S_m}(V^{\otimes m}) = SL(V)$

1985 Drinfeld Jimbo $U_q(\mathfrak{sl}_n) \curvearrowright V_q^{\otimes m} \curvearrowleft \mathcal{H}_m(q)$ 1984 Iwahory (2)

$$\mathrm{End}_{U_q(\mathfrak{sl}_n)}\left(V_q^{\otimes m}\right) \simeq \mathcal{H}_m(q)$$

2009 Rouquier $\mathcal{U}(\mathfrak{sl}_n) \curvearrowright \mathbb{V}^{\otimes m} \curvearrowleft \mathcal{B}_m$ Soergel 1992

2-Kac-Moody alg.

bimódulos de Soergel

$$\mathrm{End}_{\mathcal{U}(\mathfrak{sl}_n)}\left(\mathbb{V}^{\otimes m}\right) \cong \mathcal{B}_m$$

Elias, Williamson

Hojas ligeras

Def: Un sistema de Coxeter (W, S) es un par W grupo $S \subseteq W$, W admite una presentación por generadores $s \in S$ y rel: $s^2 = id \quad \forall s \in S$

$$\underbrace{srs\ldots}_{m(s,r)} = \underbrace{rsr\ldots}_{m(s,r)} \quad \forall r, s \in S$$

Def El álg. de Hecke $\mathcal{H} = \mathcal{H}(W,S)$ es la $\mathbb{Z}[v, v^{-1}]$-álg. $\{H_s\}_{s \in S}$

$$* \; H_s^2 = (v^{-1} - v) H_s + 1 \quad \forall s \in S$$

$(v^2 = q)$ $\quad H_s H_r H_s \cdots = \underbrace{H_r H_s H_r \cdots}_{m(s,r)}$ (4)

Para cada $x \in W$, $x = s_1 \cdots s_n$ reducida

$H_x := H_{s_1} \cdots H_{s_n}$, $\{H_x\}_{x \in W}$ es una $\mathbb{Z}[v, v^{-1}]$-base

$H_s(H_s + (v - v^{-1})) = H_s^2 + H_s(v - v^{-1})$

$= (v^{-1} - v)H_s + 1 + H_s(v - v^{-1}) = 1$

Def:
$$d : \mathcal{H} \longrightarrow \mathcal{H}$$
$$v \longmapsto v^{-1}$$
$$H_x \longmapsto (H_{x^{-1}})^{-1}$$

Teorema (Kazhdan–Lusztig)

Existe una única base $\{c'_x\}_{x \in X}$ (5) de $\mathcal{H}_6$ satisfaciendo

1) $d(c'_x) = c'_x$

2) $c'_x \in H_x + \sum_{y \in W} v\mathbb{Z}[v] H_y$

$$c'_x = H_x + \sum h_{y,x} H_y$$

Conj (KL'79) $\underbrace{h_{y,x}}_{\text{pol. de K-L.}} \in v\mathbb{N}[v]$

↳ solución Ellias–Williamson

hojas livianas

⑯

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$

$\lambda \in \mathfrak{h}^*$, $M(\lambda)$ $\qquad [M(\lambda) : L(\mu)] = 1$

$\mu \in \mathfrak{h}^*$, $L(\mu)$

[$\underline{\text{Conj}}$: $[M(w\cdot\lambda) : L(w'\cdot\lambda)] = h_{w,w'}(1)$]

(probada)

$W = \mathfrak{S}_n$, $R = \mathbb{R}[x_1, \dots, x_n]$

$\mathfrak{S}_n \circlearrowright$

$R = \bigoplus_{i \in \mathbb{Z}} R_i$, $x_i \in R_2$, $R_j = 0$

j impar

$\Theta_s = R \otimes_{R^S} R$, $s \in S$ $\langle \mathfrak{h}, \mathcal{S} = \{s\} \rangle$

$\mathcal{F}$

Def $M = \bigoplus_{i \in \mathbb{Z}} M_i$ $M(n)_i := M_{n+i}$

Def: $\mathcal{B}$ bimódulos de Soergel objetos (R,R)-bim $\mathbb{Z}$-graduados que son sumas directas de factores directos de objetos de tipo $\Theta_s \otimes_R \Theta_r \otimes_R \cdots \otimes_R \Theta_t$

Hojas Ligeras Para cada

$x \notin W$ & fijemos $S = (s_1 \cdots s_n)$ de x

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$M_x = \Theta_{s_1} - \Theta_{s_n}$

$$\begin{tikzcd}
 & & M_x \arrow[dl] \arrow[dr] & & \\
 & M_a \arrow[d, dotted] & & M_b \arrow[d, dotted] & \\
 & M_c & & M_c & \\
M_c & & & & M_c \\
 & & M_y \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

$c \in W$

Teorema (L) ⑨

Hom (M_x, M_y) admite LL como R-base a la izq. $\boxtimes$

P: $GL_n(\overline{\mathbb{F}_q})$ repres. racionales irred. sobre $\overline{\mathbb{F}_q}$.

Ivan Shestakov (USP)

A.I. Maltsev

$\mathcal{M}$, $\mathcal{M}_n$ = variedad de álgebras de um tipo gen. por n elem.

$\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \dots \subseteq \mathcal{M}_n$, $\mathcal{M} = \bigcup \mathcal{M}_n$

$r_b(\mathcal{M}) = \min\{n \mid \mathcal{M} = \mathcal{M}_n\}$

$r_b(As) = 2$

$r_b(Lie) = 2$

$r_b(Var\, G) = \infty$

$r_b([x,y]^2 = 0) = \infty$

$r_b(x^n = 0, \text{char } k = p \geq n) = \infty$

Iván Angiono

Trab. DeConcini, Procesi, Reshetikhin.

$\mathfrak{g}$ álg de Lie semisimple

$A = (a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$

$(d_1, \dots, d_\theta) \in \mathbb{N}^\theta \,/\, (d_i a_{ij})$ sim

<u>Def</u> $U_q(\mathfrak{g})$ es el álg. presentada por generadores $E_i, F_i, K_i^{\pm 1}$, $1 \leq i \leq \theta$

$K_i K_i^{-1} = 1, \quad K_i K_j = K_j K_i$

$K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j$

$E_i F_j - F_j E_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad q_i = q^{d_i}$

$$(ad_c E_i)^{1-a_{ij}} E_j = 0 = (ad_c F_i)^{1-a_{ij}} F_j = 0 \quad (2)$$

Hechos

- $U_q(\mathfrak{g})$ es un álg. de Hopf

  $\Delta(K_i) = K_i \otimes K_i$

  $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$

  $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes K_i$

- $U_q(\mathfrak{g})$ es $\mathbb{Z}^{\Theta}$-graduada

  $\{\alpha_i\}$ base canónica

  $|E_i| = \alpha_i = -|F_i|, \; |K_i| = 0$

- $U_q^+ =$ subálg gen. por E_i's
- $U_q^- =$ – – – F_i's

U_q^0 = subalg. gen por k_i's ③

$U_q^+ \otimes U_q^0 \otimes U_q^- \xrightarrow{\sim} U_q$ desc. triang.

- $\Delta = \Delta_+ \cup \Delta_-$ conj. de raíces

- W = grupo de Weyl.

generado por $s_i : Q \to Q$

$\{\alpha_i\}$ base de raíces simples

$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad \forall 1 \leq i, j \leq \theta$

$w_0 \in W$ elemento de longitud máxima.

$w_0 = s_{i_1} \cdots s_{i_N}$ exp. reducida 4

$\underbrace{s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})}_{= \beta_j} \in \Delta_+$

$\beta_j \neq \beta_k$, $\{\beta_j\} = \Delta_+$

Isomorfismos de Lusztig

$T_i : U_q \longrightarrow U_q$

$T_i(E_i) = -F_i K_i$, $T_i(F_i) = -K_i^{-1} E_i$

$T_i(E_j) = (\mathrm{ad}_c E_i)^{-a_{ij}} E_j$

$T_i(F_j) = (\mathrm{ad}_c F_i)^{-a_{ij}} F_j$

$T_i(K_j) = K_j K_i^{-a_{ij}}$

$T_i((U_q)_\alpha) = (U_q)_{s_i(\alpha)}$

$E_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(E_j) \in (U_q)_{\beta_j}$ Ⓢ

Prop $\{E_{\beta_1}^{n_1} \cdots E_{\beta_N}^{n_N} : n_i \in \mathbb{N}\}$ base de U_q^+

$\Rightarrow \{E_{\beta_1}^{n_1} \cdots E_{\beta_N}^{n_N} K_1^{h_1} \cdots K_\theta^{h_\theta} F_{\beta_1}^{m_1} \cdots F_{\beta_N}^{m_N}\}$ base de U_q ∎

$\boxed{k<j}$

$E_{\beta_k} E_{\beta_j} - q_{\beta_k \beta_j} E_{\beta_j} E_{\beta_k} = \sum$ prod en E_{β_i}'s $j>i>k$

Fijemos $q^\ell = 1$

Prop (de Concini, Procesi)

$$E_{\beta_k}^{\ell},\ F_{\beta_k}^{\ell},\ K_i^{\ell} \in Z(U_q) \quad (*)$$

$Z_0 =$ subálg. generada por $(*)$

- es una subálg. de Hopf.

$\square$

$$\begin{tikzcd}
Z_0 \arrow[rr, hook] \arrow[dr, hook] & & U_q \\
& Z_q \arrow[ur, hook] &
\end{tikzcd}$$

$Z_q =$ centro de U_q

$$\begin{tikzcd}
\operatorname{Spec} Z_0 & & \operatorname{Irr} U_q \arrow[ll] \arrow[dl] & \circledast \\
& \operatorname{Spec} Z_q \arrow[ul, "\pi"] & &
\end{tikzcd}$$

$P \in \operatorname{Spec} Z_q \Rightarrow \exists!$ irrep "genérica" de dimensión $\ell^{|\Delta_+|}$ de U_q

$\rightsquigarrow V_P$

Teorema [dCPR]

$x = \pi(P)$, $y = \pi(Q)$

$$V_P \otimes V_Q \cong \bigoplus_{R \in \pi^{-1}(xy)} V_R^{\oplus(|\Delta_+| - \theta)}$$

Baby Verma Modules

$\mathbb{C}_\lambda = \mathbb{C}v$, $\lambda \in \widehat{U_q^0}$

$k \cdot v = \lambda(k) v$, $e_i v = 0$

$F_\beta^\ell v = b_\beta v \qquad b = \{b_\alpha\}_{\alpha \in \Delta_+}$

$\overline{M}(\lambda, b) = U_q \otimes_S \mathbb{C}_{\lambda, b}$

$b = 0 \rightsquigarrow$ pasa a una rep de u_q

Álgebras con traza

(R, t) , $t : R \to R$ $\mathbb{C}$-lineal

tq a) $t(a)b = a\,t(b)$ $\forall a, b$

b) $t(ab) = t(ba)$, $\forall a, b$ 9

c) $t(t(a)b) = t(a)t(b)$

$t(R)$ álgebra traza

$M \in \mathbb{C}^{n \times n}$

$$\chi_M(t) = \det(t - M)$$
$$= t^n + \sum_{i=1}^{n} (-1)^i P_i(\mathrm{tr} M, \dots, \mathrm{tr} M^i) t^{n-i}$$

<u>Def</u> (R, t) es n-Cayley-Hamilton si $\forall a \in R$

$$\chi_a(X) = X^n + \sum (-1)^i P_i(t(a), \dots, t(a^i)) X^{n-i}$$

$\chi_a(0) = 0$, $\chi_a(1) = n$ 110

$A = t(R)$, B una A-àlg commut

$\Rightarrow R \otimes_A B$ es n-Cayley Hami!